Chapter 13

Maxwell's Equations and Boundary Conditions

13.1 INTRODUCTION

The behavior of the electric field intensity **E** and the electric flux density **D** across the interface of two different materials was examined in Chapter 7, where the fields were static. A similar treatment will now be given for the magnetic field strength **H** and the magnetic flux density **B**, again with static fields. This will complete the study of the boundary conditions on the four principal vector fields.

In Chapter 12, where time-variable fields were treated, displacement current density $\mathbf{J}_D$ was introduced and Faraday's law was examined. In this chapter these same equations and others developed earlier are grouped together to form the set known as *Maxwell's equations*. These equations underlie all of electromagnetic field theory; they should be memorized.

13.2 BOUNDARY RELATIONS FOR MAGNETIC FIELDS

When **H** and **B** are examined at the interface between two different materials, abrupt changes can be expected, similar to those noted in **E** and **D** at the interface between two different dielectrics (see Section 7.7).

In Fig. 13-1 an interface is shown separating material *1*, with properties σ_1 and μ_{r1}, from *2*, with σ_2 and μ_{r2}. The behavior of **B** can be determined by use of a small right circular cylinder positioned across the interface as shown. Since magnetic flux lines are continuous,

$$\oint \mathbf{B}\cdot d\mathbf{S} = \int_{\text{end }1} \mathbf{B}_1 \cdot d\mathbf{S}_1 + \int_{\text{cyl}} \mathbf{B}\cdot d\mathbf{S} + \int_{\text{end }2} \mathbf{B}_2 \cdot d\mathbf{S}_2 = 0$$

Now if the two planes are allowed to approach one another, keeping the interface between them, the area of the curved surface will approach zero, giving

$$\int_{\text{end }1} \mathbf{B}_1 \cdot d\mathbf{S}_1 + \int_{\text{end }2} \mathbf{B}_2 \cdot d\mathbf{S}_2 = 0$$

or

$$-B_{n1}\int_{\text{end }1} dS_1 + B_{n2}\int_{\text{end }2} dS_2 = 0$$

from which

$$B_{n1} = B_{n2}$$

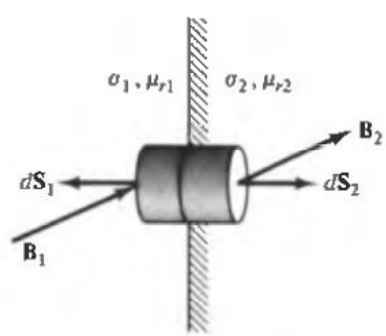

Fig. 13-1

In words, *the normal component of* **B** *is continuous across an interface.* Note that *either* normal to the interface may be used in calculating B_{n1} and B_{n2}.

The variation in **H** across an interface is obtained by the application of Ampère's law around a closed rectangular path, as shown in Fig. 13-2. Assuming no current at the interface, and letting the rectangle shrink to zero in the usual way,

$$0 = \oint \mathbf{H} \cdot d\mathbf{l} \rightarrow H_{\ell 1}\Delta \ell_1 - H_{\ell 2}\Delta \ell_2$$

whence

$$H_{\ell 1} = H_{\ell 2}$$

Thus tangential **H** has the same projection along the two sides of the rectangle. Since the rectangle can be rotated 90° and the argument repeated, it follows that

$$H_{t1} = H_{t2}$$

In words, *the tangential component of* **H** *is continuous across a current-free interface.*

The relation

$$\frac{\tan \theta_1}{\tan \theta_2} = \frac{\mu_{r2}}{\mu_{r1}}$$

between the angles made by $\mathbf{H}_1$ and $\mathbf{H}_2$ with a current-free interface (see Fig. 13-2) is obtained by analogy with Example 6, Section 7.7.

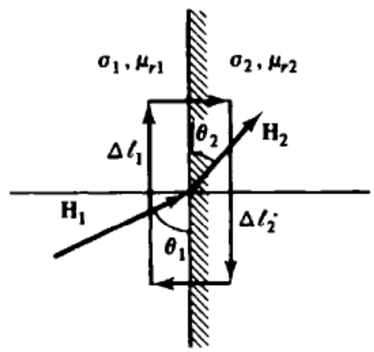

Fig. 13-2

13.3 CURRENT SHEET AT THE BOUNDARY

If one material at the interface has a nonzero conductivity, a current may be present. This could be a current throughout the material; however, of more interest is the case of a current sheet at the interface.

Figure 13-3 shows a uniform current sheet. In the indicated coordinate system the current sheet has density $\mathbf{K} = K_0\mathbf{a}_y$ and it is located at the interface $x = 0$ between regions *1* and *2*. The magnetic field $\mathbf{H}'$ produced by this current sheet is given by Example 2, Section 9.2,

$$\mathbf{H}'_1 = \tfrac{1}{2}\mathbf{K} \times \mathbf{a}_{n1} = \tfrac{1}{2}K_0\mathbf{a}_z \qquad \mathbf{H}'_2 = \tfrac{1}{2}\mathbf{K} \times \mathbf{a}_{n2} = \tfrac{1}{2}K_0(-\mathbf{a}_z)$$

Thus $\mathbf{H}'$ has a tangential discontinuity of magnitude $|K_0|$ at the interface. If a second magnetic field, $\mathbf{H}''$, arising from some other source, is present, its tangential component will be continuous at the interface. The resultant magnetic field,

$$\mathbf{H} = \mathbf{H}' + \mathbf{H}''$$

will then have a discontinuity of magnitude $|K_0|$ in its tangential component. This is expressed by the vector formula

$$(\mathbf{H}_1 - \mathbf{H}_2) \times \mathbf{a}_{n12} = \mathbf{K}$$

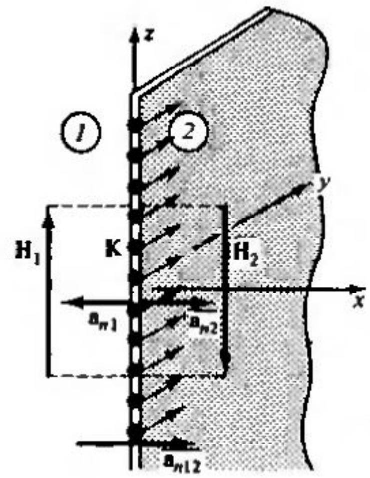

Fig. 13-3

where $\mathbf{a}_{n12}$ is the unit normal from region *1* to region *2*. The vector relation, which is independent of the choice of coordinate system, also holds for a nonuniform current sheet, where **K** is the value of the current density at the considered point of the interface.

13.4 SUMMARY OF BOUNDARY CONDITIONS

For reference purposes, the relationships for **E** and **D** across the interface of two dielectrics are shown below along with the relationships for **H** and **B**.

Magnetic Fields

$$B_{n1} = B_{n2}$$

$$\begin{cases} H_{t1} = H_{t2} & \text{(current-free)} \\ (\mathbf{H}_1 - \mathbf{H}_2) \times \mathbf{a}_{n12} = \mathbf{K} & \text{(with current sheet)} \end{cases}$$

$$\frac{\tan\theta_1}{\tan\theta_2} = \frac{\mu_{r2}}{\mu_{r1}} \quad \text{(current-free)}$$

Electric Fields

$$\begin{cases} D_{n1} = D_{n2} & \text{(charge-free)} \\ (\mathbf{D}_1 - \mathbf{D}_2) \cdot \mathbf{a}_{n12} = -\rho_s & \text{(with surface charge)} \end{cases}$$

$$E_{t1} = E_{t2}$$

$$\frac{\tan\theta_1}{\tan\theta_2} = \frac{\epsilon_{r2}}{\epsilon_{r1}} \quad \text{(charge-free)}$$

These relationships were obtained assuming static conditions. However, in Chapter 14 they will be found to apply equally well to time-variable fields.

13.5 MAXWELL'S EQUATIONS

A static **E** field can exist in the absence of a magnetic field **H**; a capacitor with a static charge Q furnishes an example. Likewise, a conductor with a constant current I has a magnetic field **H** without an **E** field. When fields are time-variable, however, **H** cannot exist without an E field nor can **E** exist without a corresponding **H** field. While much valuable information can be derived from static field theory, only with time-variable fields can the full value of electromagnetic field theory be demonstrated. The experiments of Faraday and Hertz and the theoretical analyses of Maxwell all involved time-variable fields.

The equations grouped below, called *Maxwell's equations,* were separately developed and examined in earlier chapters. In Table 13-1, the most general form is presented, where charges and conduction current may be present in the region. Note that the point and integral forms of the first two equations are equivalent under Stokes' theorem, while the point and integral forms of the last two equations are equivalent under the divergence theorem.

Table 13-1. Maxwell's Equations, General Set

Point Form	Integral Form
$\nabla \times \mathbf{H} = \mathbf{J}_c + \frac{\partial \mathbf{D}}{\partial t}$	$\oint \mathbf{H} \cdot d\mathbf{l} = \int_S \left(\mathbf{J}_c + \frac{\partial \mathbf{D}}{\partial t}\right) \cdot d\mathbf{S}$ (Ampère's law)
$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$	$\oint \mathbf{E} \cdot d\mathbf{l} = \int_S \left(-\frac{\partial \mathbf{B}}{\partial t}\right) \cdot d\mathbf{S}$ (Faraday's law; S fixed)
$\nabla \cdot \mathbf{D} = \rho$	$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_v \rho \, dv$ (Gauss' law)
$\nabla \cdot \mathbf{B} = 0$	$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0$ (nonexistence of monopole)

For free space, where there are no charges ($\rho = 0$) and no conduction currents ($\mathbf{J}_c = 0$), Maxwell's equations take the form shown in Table 13-2.

Table 13-2. Maxwell's Equations, Free-Space Set

Point Form	Integral Form
$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t}$	$\oint \mathbf{H} \cdot d\mathbf{l} = \int_S \left(\frac{\partial \mathbf{D}}{\partial t}\right) \cdot d\mathbf{S}$
$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$	$\oint \mathbf{E} \cdot d\mathbf{l} = \int_S \left(-\frac{\partial \mathbf{B}}{\partial t}\right) \cdot d\mathbf{S}$
$\nabla \cdot \mathbf{D} = 0$	$\oint_S \mathbf{D} \cdot d\mathbf{S} = 0$
$\nabla \cdot \mathbf{B} = 0$	$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0$

The first and second point-form equations in the free-space set can be used to show that time-variable **E** and **H** fields cannot exist independently. For example, if **E** is a function of time, then $\mathbf{D} = \epsilon_0 \mathbf{E}$ will also be a function of time, so that $\partial \mathbf{D}/\partial t$ will be nonzero. Consequently, $\nabla \times \mathbf{H}$ is nonzero, and so a nonzero **H** must exist. In a similar way, the second equation can be used to show that if **H** is a function of time, then there must be an **E** field present.

The point form of Maxwell's equations is used most frequently in the problems. However, the integral form is important in that it better displays the underlying physical laws.

Solved Problems

13.1. In region *1* of Fig. 13-4, $\mathbf{B}_1 = 1.2\mathbf{a}_x + 0.8\mathbf{a}_y + 0.4\mathbf{a}_z$ (T). Find $\mathbf{H}_2$ (i.e., **H** at $z = +0$) and the angles between the field vectors and a tangent to the interface

Write $\mathbf{H}_1$ directly below $\mathbf{B}_1$. Then write those components of $\mathbf{H}_2$ and $\mathbf{B}_2$ which follow directly from the two rules **B** *normal is continuous* and **H** *tangential is continuous* across a current-free interface.

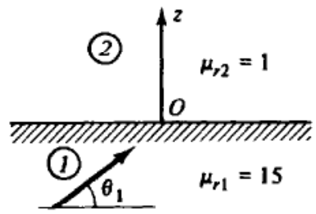

Fig. 13-4

$$\mathbf{B}_1 = 1.2\mathbf{a}_x + 0.8\mathbf{a}_y + 0.4\mathbf{a}_z \quad \text{(T)}$$

$$\mathbf{H}_1 = \frac{1}{\mu_0}(8.0\mathbf{a}_x + 5.33\mathbf{a}_y + 2.67\mathbf{a}_z)10^{-2} \quad \text{(A/m)}$$

$$\mathbf{H}_2 = \frac{1}{\mu_0}(8.0\mathbf{a}_x + 5.33\mathbf{a}_y + 10^2\mu_0 H_{z2}\mathbf{a}_z)10^{-2} \quad \text{(A/m)}$$

$$\mathbf{B}_2 = B_{x2}\mathbf{a}_x + B_{y2}\mathbf{a}_y + 0.4\mathbf{a}_z \quad \text{(T)}$$

Now the remaining terms follow directly:

$$B_{x2} = \mu_0\mu_{r2}H_{x2} = 8.0 \times 10^{-2}\,\text{(T)} \qquad B_{y2} = 5.33 \times 10^{-2}\,\text{(T)} \qquad H_{z2} = \frac{B_{z2}}{\mu_0\mu_{r2}} = \frac{0.4}{\mu_0}\,\text{(A/m)}$$

Angle θ_1 is $90° - \alpha_1$, where α_1 is the angle between $\mathbf{B}_1$ and the normal, $\mathbf{a}_z$.

$$\cos\alpha_1 = \frac{\mathbf{B}_1 \cdot \mathbf{a}_z}{|\mathbf{B}_1|} = 0.27$$

whence $\alpha_1 = 74.5°$ and $\theta_1 = 15.5°$. Similarly, $\theta_2 = 76.5°$.

Check: $(\tan\theta_1)/(\tan\theta_2) = \mu_{r2}/\mu_{r1}$.

13.2. Region *1*, for which $\mu_{r1} = 3$, is defined by $x < 0$ and region *2*, $x > 0$, has $\mu_{r2} = 5$. Given

$$\mathbf{H}_1 = 4.0\mathbf{a}_x + 3.0\mathbf{a}_y - 6.0\mathbf{a}_z \quad \text{(A/m)}$$

show that $\theta_2 = 19.7°$ and that $H_2 = 7.12$ A/m.

Proceed as in Problem 13.1.

$$\mathbf{H}_1 = 4.0\mathbf{a}_x + 3.0\mathbf{a}_y - 6.0\mathbf{a}_z \quad \text{(A/m)}$$

$$\mathbf{B}_1 = \mu_0(12.0\mathbf{a}_x + 9.0\mathbf{a}_y - 18.0\mathbf{a}_z) \quad \text{(T)}$$

$$\mathbf{B}_2 = \mu_0(12.0\mathbf{a}_x + 15.0\mathbf{a}_y - 30.0\mathbf{a}_z) \quad \text{(T)}$$

$$\mathbf{H}_2 = 2.40\mathbf{a}_x + 3.0\mathbf{a}_y - 6.0\mathbf{a}_z \quad \text{(A/m)}$$

Now

$$H_2 = \sqrt{(2.40)^2 + (3.0)^2 + (-6.0)^2} = 7.12 \text{ A/m}$$

The angle α_2 between $\mathbf{H}_2$ and the normal is given by

$$\cos\alpha_2 = \frac{H_{x2}}{H_2} = 0.34 \qquad \text{or} \qquad \alpha_2 = 70.3°$$

Then $\theta_2 = 90° - \alpha_2 = 19.7°$.

13.3. Region *1*, where $\mu_{r1} = 4$, is the side of the plane $y + z = 1$ containing the origin (see Fig. 13-5). In region *2*, $\mu_{r2} = 6$. $\mathbf{B}_1 = 2.0\mathbf{a}_x + 1.0\mathbf{a}_y$ (T), find $\mathbf{B}_2$ and $\mathbf{H}_2$.

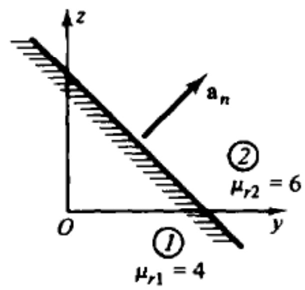

Fig. 13-5

Choosing the unit normal $\mathbf{a}_n = (\mathbf{a}_y + \mathbf{a}_z)/\sqrt{2}$,

$$B_{n1} = \frac{(2.0\mathbf{a}_x + 1.0\mathbf{a}_y)\cdot(\mathbf{a}_y + \mathbf{a}_z)}{\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$\mathbf{B}_{n1} = \left(\frac{1}{\sqrt{2}}\right)\mathbf{a}_n = 0.5\mathbf{a}_y + 0.5\mathbf{a}_z = \mathbf{B}_{n2}$$

$$\mathbf{B}_{t1} = \mathbf{B}_1 - \mathbf{B}_{n1} = 2.0\mathbf{a}_x + 0.5\mathbf{a}_y - 0.5\mathbf{a}_z$$

$$\mathbf{H}_{t1} = \frac{1}{\mu_0}(0.5\mathbf{a}_x + 0.125\mathbf{a}_y - 0.125\mathbf{a}_z) = \mathbf{H}_{t2}$$

$$\mathbf{B}_{t2} = \mu_0\mu_{r2}\mathbf{H}_{t2} = 3.0\mathbf{a}_x + 0.75\mathbf{a}_y - 0.75\mathbf{a}_z$$

Now the normal and tangential parts of $\mathbf{B}_2$ are combined.

$$\mathbf{B}_2 = 3.0\mathbf{a}_x + 1.25\mathbf{a}_y - 0.25\mathbf{a}_z \quad \text{(T)}$$

$$\mathbf{H}_2 = \frac{1}{\mu_0}(0.50\mathbf{a}_x + 0.21\mathbf{a}_y - 0.04\mathbf{a}_z) \quad \text{(A/m)}$$

13.4. In region *1*, defined by $z < 0$, $\mu_{r1} = 3$ and

$$\mathbf{H}_1 = \frac{1}{\mu_0}(0.2\mathbf{a}_x + 0.5\mathbf{a}_y + 1.0\mathbf{a}_z) \quad \text{(A/m)}$$

Find $\mathbf{H}_2$ if it is known that $\theta_2 = 45°$.

$$\cos\alpha_1 = \frac{\mathbf{H}_1 \cdot \mathbf{a}_z}{|\mathbf{H}_1|} = 0.88 \qquad \text{or} \qquad \alpha_1 = 28.3°$$

Then, $\theta_1 = 61.7°$ and

$$\frac{\tan 61.7°}{\tan 45°} = \frac{\mu_{r2}}{3} \qquad \text{or} \qquad \mu_{r2} = 5.57$$

From the continuity of normal **B**, $\mu_{r1}H_{z1} = \mu_{r2}H_{z2}$, and so

$$\mathbf{H}_2 = \frac{1}{\mu_0}\left(0.2\mathbf{a}_x + 0.5\mathbf{a}_y + \frac{\mu_{r1}}{\mu_{r2}}1.0\mathbf{a}_z\right) = \frac{1}{\mu_0}(0.2\mathbf{a}_x + 0.5\mathbf{a}_y + 0.54\mathbf{a}_z) \quad \text{(A/m)}$$

13.5. A current sheet, $\mathbf{K} = 6.5\mathbf{a}_z$ A/m, at $x = 0$ separates region *1*, $x < 0$, where $\mathbf{H}_1 = 10\mathbf{a}_y$ A/m and region *2*, $x > 0$. Find $\mathbf{H}_2$ at $x = +0$.

Nothing is said about the permeabilities of the two regions; however, since $\mathbf{H}_1$ is entirely tangential, a change in permeability would have no effect. Since $B_{n1} = 0$, $B_{n2} = 0$ and therefore $H_{n2} = 0$.

$$(\mathbf{H}_1 - \mathbf{H}_2) \times \mathbf{a}_{n12} = \mathbf{K}$$
$$(10\mathbf{a}_y - H_{y2}\mathbf{a}_y) \times \mathbf{a}_x = 6.5\mathbf{a}_z$$
$$(10 - H_{y2})(-\mathbf{a}_z) = 6.5\mathbf{a}_z$$
$$H_{y2} = 16.5 \text{ (A/m)}$$

Thus, $\mathbf{H}_2 = 16.5\mathbf{a}_y$ (A/m).

13.6. A current sheet, $\mathbf{K} = 9.0\mathbf{a}_y$ A/m, is located at $z = 0$, the interface between region *1*, $z < 0$, with $\mu_{r1} = 4$, and region *2*, $z > 0$, $\mu_{r2} = 3$. Given that $\mathbf{H}_2 = 14.5\mathbf{a}_x + 8.0\mathbf{a}_z$ (A/m), find $\mathbf{H}_1$.

Mathcad

The current sheet shown in Fig. 13-6 is first examined alone.

$$\mathbf{H}_1' = \tfrac{1}{2}(9.0)\mathbf{a}_y \times (-\mathbf{a}_z) = 4.5(-\mathbf{a}_x)$$
$$\mathbf{H}_2' = \tfrac{1}{2}(9.0)\mathbf{a}_y \times \mathbf{a}_z = 4.5\mathbf{a}_x$$

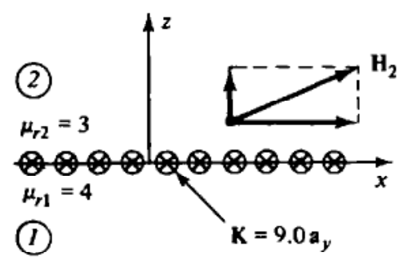

Fig. 13-6

From region *1* to region *2*, H_x will increase by 9.0 A/m due to the current sheet.

Now the complete **H** and **B** fields are examined.

$$\mathbf{H}_2 = 14.5\mathbf{a}_x + 8.0\mathbf{a}_z \quad \text{(A/m)}$$
$$\mathbf{B}_2 = \mu_0(43.5\mathbf{a}_x + 24.0\mathbf{a}_z) \quad \text{(T)}$$
$$\mathbf{B}_1 = \mu_0(22.0\mathbf{a}_x + 24.0\mathbf{a}_z) \quad \text{(T)}$$
$$\mathbf{H}_1 = 5.5\mathbf{a}_x + 6.0\mathbf{a}_z \quad \text{(A/m)}$$

Note that H_{x1} must be 9.0 A/m less than H_{x2} because of the current sheet. B_{x1} is obtained as $\mu_0\mu_{r1}H_{x1}$.

An alternate method is to apply $(\mathbf{H}_1 - \mathbf{H}_2) \times \mathbf{a}_{n12} = \mathbf{K}$:

$$(H_{x1}\mathbf{a}_x + H_{y1}\mathbf{a}_y + H_{z1}\mathbf{a}_z) \times \mathbf{a}_z = \mathbf{K} + (14.5\mathbf{a}_x + 8.0\mathbf{a}_z) \times \mathbf{a}_z$$
$$-H_{x1}\mathbf{a}_y + H_{y1}\mathbf{a}_x = -5.5\mathbf{a}_y$$

from which $H_{x1} = 5.5$ A/m and $H_{y1} = 0$. This method deals exclusively with tangential **H**; any normal component must be determined by the previous methods.

13.7. Region *1*, $z < 0$, has $\mu_{r1} = 1.5$, while region *2*, $z > 0$, has $\mu_{r2} = 5$. Near (0, 0, 0),

$$\mathbf{B}_1 = 2.40\mathbf{a}_x + 10.0\mathbf{a}_z \quad \text{(T)} \qquad \mathbf{B}_2 = 25.75\mathbf{a}_x - 17.7\mathbf{a}_y + 10.0\mathbf{a}_z \quad \text{(T)}$$

If the interface carries a sheet current, what is its density at the origin?

Near the origin,

$$\mathbf{H}_1 = \frac{1}{\mu_0\mu_{r1}}\mathbf{B}_1 = \frac{1}{\mu_0}(1.60\mathbf{a}_x + 6.67\mathbf{a}_z) \quad \text{(A/m)}$$
$$\mathbf{H}_2 = \frac{1}{\mu_0}(5.15\mathbf{a}_x - 3.54\mathbf{a}_y + 2.0\mathbf{a}_z) \quad \text{(A/m)}$$

Then the local value of **K** is given by

$$\mathbf{K} = (\mathbf{H}_1 - \mathbf{H}_2) \times \mathbf{a}_{n12} = \frac{1}{\mu_0}(-3.55\mathbf{a}_x + 3.54\mathbf{a}_y + 4.67\mathbf{a}_z) \times \mathbf{a}_z = \frac{5.0}{\mu_0}\left(\frac{\mathbf{a}_x + \mathbf{a}_y}{\sqrt{2}}\right) \quad \text{(A/m)}$$

13.8. Given $\mathbf{E} = E_m \sin(\omega t - \beta z)\mathbf{a}_y$ in free space, find **D**, **B** and **H**. Sketch **E** and **H** at $t = 0$.

$$\mathbf{D} = \epsilon_0 \mathbf{E} = \epsilon_0 E_m \sin(\omega t - \beta z)\mathbf{a}_y$$

The Maxwell equation $\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t$ gives

$$\begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 0 & E_m \sin(\omega t - \beta z) & 0 \end{vmatrix} = -\frac{\partial \mathbf{B}}{\partial t}$$

or

$$-\frac{\partial \mathbf{B}}{\partial t} = \beta E_m \cos(\omega t - \beta z)\mathbf{a}_x$$

Integrating,

$$\mathbf{B} = -\frac{\beta E_m}{\omega} \sin(\omega t - \beta z)\mathbf{a}_x$$

where the "constant" of integration, which is a static field, has been neglected. Then,

$$\mathbf{H} = -\frac{\beta E_m}{\omega \mu_0} \sin(\omega t - \beta z)\mathbf{a}_x$$

Note that **E** and **H** are mutually perpendicular. At $t = 0$, $\sin(\omega t - \beta z) = -\sin \beta z$. Figure 13-7 shows the two fields along the z axis, on the assumption that E_m and β are positive.

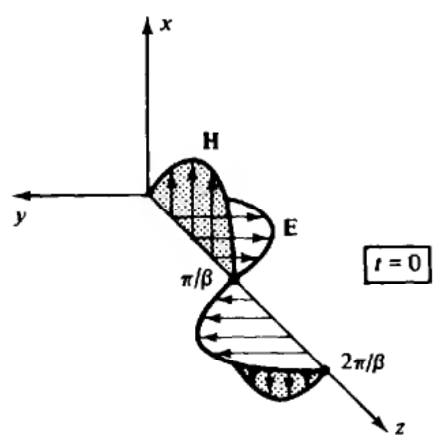

Fig. 13-7

13.9. Show that the **E** and **H** fields of Problem 13.8 constitute a wave traveling in the z direction. Verify that the wave speed and E/H depend only on the properties of free space.

E and **H** together vary as $\sin(\omega t - \beta z)$. A given state of **E** and **H** is then characterized by

$$\omega t - \beta z = \text{const.} = \omega t_0 \qquad \text{or} \qquad z = \frac{\omega}{\beta}(t - t_0)$$

But this is the equation of a plane moving with speed

$$c = \frac{\omega}{\beta}$$

in the direction of its normal, $\mathbf{a}_z$. (It is assumed that β, as well as ω, is positive; for β negative, the direction of motion would be $-\mathbf{a}_z$.) Thus, the entire pattern of Fig. 13-7 moves down the z axis with speed c.

The Maxwell equation $\nabla \times \mathbf{H} = \partial \mathbf{D}/\partial t$ gives

$$\begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{-\beta E_m}{\omega\mu_0}\sin(\omega t - \beta z) & 0 & 0 \end{vmatrix} = \frac{\partial}{\partial t}\epsilon_0 E_m \sin(\omega t - \beta z)\mathbf{a}_y$$

$$\frac{\beta^2 E_m}{\omega\mu_0}\cos(\omega t - \beta z)\mathbf{a}_y = \epsilon_0 E_m \omega \cos(\omega t - \beta z)\mathbf{a}_y$$

$$\frac{1}{\epsilon_0\mu_0} = \frac{\omega^2}{\beta^2}$$

Consequently,

$$c = \sqrt{\frac{1}{\epsilon_0\mu_0}} \approx \sqrt{\frac{1}{(10^{-9}/36\pi)(4\pi \times 10^{-7})}} = 3 \times 10^8 \text{ (m/s)}$$

Moreover,

$$\frac{E}{H} = \frac{\omega\mu_0}{\beta} = \sqrt{\frac{\mu_0}{\epsilon_0}} \approx 120\pi \text{ (V/A)} = 120\pi\ \Omega$$

13.10. Given $\mathbf{H} = H_m e^{j(\omega t + \beta z)}\mathbf{a}_x$ in free space, find $\mathbf{E}$.

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t}$$

$$\frac{\partial}{\partial z} H_m e^{j(\omega t + \beta z)}\mathbf{a}_y = \frac{\partial \mathbf{D}}{\partial t}$$

$$j\beta H_m e^{j(\omega t + \beta z)}\mathbf{a}_y = \frac{\partial \mathbf{D}}{\partial t}$$

$$\mathbf{D} = \frac{\beta H_m}{\omega} e^{j(\omega t + \beta z)}\mathbf{a}_y$$

and $\mathbf{E} = \mathbf{D}/\epsilon_o$.

13.11. Given

$$\mathbf{E} = 30\pi e^{j(10^8 t + \beta z)}\mathbf{a}_x \text{ (V/m)} \qquad \mathbf{H} = H_m e^{j(10^8 t + \beta z)}\mathbf{a}_y \text{ (A/m)}$$

in free space, find H_m and B $(\beta > 0)$.

This is a plane wave, essentially the same as that in Problems 13.8 and 13.9 (except that, there, $\mathbf{E}$ was in the y direction and $\mathbf{H}$ in the x direction). The results of Problem 13.9 hold for any such wave in free space:

$$\frac{\omega}{\beta} = \frac{1}{\sqrt{\epsilon_0\mu_0}} = 3 \times 10^8 \text{ (m/s)} \qquad \frac{E}{H} = \sqrt{\frac{\mu_0}{\epsilon_0}} = 120\pi\ \Omega$$

Thus, for the given wave,

$$\beta = \frac{10^8}{3 \times 10^8} = \frac{1}{3} \text{ (rad/m)} \qquad H_m = \pm\frac{30\pi}{120\pi} = \pm\frac{1}{4} \text{ (A/m)}$$

To fix the sign of H_m, apply $\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t$:

$$j\beta 30\pi e^{j(10^8 t + \beta z)}\mathbf{a}_y = -j10^8 \mu_0 H_m e^{j(10^8 t + \beta z)}\mathbf{a}_y$$

which shows that H_m must be negative.

13.12. In a homogeneous nonconducting region where $\mu_r = 1$, find ϵ_r and ω if

$$\mathbf{E} = 30\pi e^{j[\omega t-(4/3)y]}\mathbf{a}_z \quad \text{(V/m)} \qquad \mathbf{H} = 1.0e^{j[\omega t-(4/3)y]}\mathbf{a}_x \quad \text{(A/m)}$$

Here, by analogy to Problem 13.9,

$$\frac{\omega}{\beta} = \frac{1}{\sqrt{\epsilon\mu}} = \frac{3\times 10^8}{\sqrt{\epsilon_r\mu_r}} \quad \text{(m/s)} \qquad \frac{E}{H} = \sqrt{\frac{\mu}{\epsilon}} = 120\pi\sqrt{\frac{\mu_r}{\epsilon_r}} \quad (\Omega)$$

Thus, since $\mu_r = 1$,

$$\frac{\omega}{\frac{4}{3}} = \frac{3\times 10^8}{\sqrt{\epsilon_r}} \qquad 30\pi = 120\pi\frac{1}{\sqrt{\epsilon_r}}$$

which yield $\epsilon_r = 16$, $\omega = 10^8$ rad/s. In this medium the speed of light is $c/4$.

Supplementary Problems

13.13. Region *1*, where $\mu_{r1} = 5$, is on the side of the plane $6x + 4y + 3z = 12$ that includes the origin. In region *2*, $\mu_{r2} = 3$. Given

$$\mathbf{H}_1 = \frac{1}{\mu_0}(3.0\mathbf{a}_x - 0.5\mathbf{a}_y) \quad \text{(A/m)}$$

find $\mathbf{B}_2$ and θ_2. *Ans.* $12.15\mathbf{a}_x + 0.60\mathbf{a}_y + 1.58\mathbf{a}_z$ (T), 56.6°

13.14. The interface between two different regions is normal to one of the three cartesian axes. If

$$\mathbf{B}_1 = \mu_0(43.5\mathbf{a}_x + 24.0\mathbf{a}_z) \qquad \mathbf{B}_2 = \mu_0(22.0\mathbf{a}_x + 24.0\mathbf{a}_z)$$

what is the ratio $(\tan\theta_1)/(\tan\theta_2)$? *Ans.* 0.506

13.15. Inside a right circular cylinder, $\mu_{r1} = 1000$. The exterior is free space. If $\mathbf{B}_1 = 2.5\mathbf{a}_\phi$ (T) inside the cylinder, determine $\mathbf{B}_2$ just outside. *Ans.* $2.5\mathbf{a}_\phi$ (mT)

13.16. In spherical coordinates, region *1* is $r < a$, region *2* is $a < r < b$ and region *3* is $r > b$. Regions *1* and *3* are free space, while $\mu_{r2} = 500$. Given $\mathbf{B}_1 = 0.20\mathbf{a}_r$ (T), find $\mathbf{H}$ in each region.
Ans. $\frac{0.20}{\mu_0}$ (A/m), $\frac{4\times 10^{-4}}{\mu_0}$ (A/m), $\frac{0.20}{\mu_0}$ (A/m)

13.17. A current sheet, $\mathbf{K} = (8.0/\mu_0)\mathbf{a}_y$ (A/m), at $x = 0$ separates region *1*, $x < 0$ and $\mu_{r1} = 3$, from region *2*, $x > 0$ and $\mu_{r2} = 1$. Given $\mathbf{H}_1 = (10.0/\mu_0)(\mathbf{a}_y + \mathbf{a}_z)$ (A/m), find $\mathbf{H}_2$.
Ans. $\frac{1}{\mu_0}(10.0\mathbf{a}_y + 2.0\mathbf{a}_z)$ (A/m)

13.18. The $x = 0$ plane contains a current sheet of density $\mathbf{K}$ which separates region *1*, $x < 0$ and $\mu_{r1} = 2$, from region *2*, $x < 0$ and $\mu_{r2} = 7$. Given

$$\mathbf{B}_1 = 6.0\mathbf{a}_x + 4.0\mathbf{a}_y + 10.0\mathbf{a}_z \quad \text{(T)} \qquad \mathbf{B}_2 = 6.0\mathbf{a}_x - 50.96\mathbf{a}_y + 8.96\mathbf{a}_z \quad \text{(T)}$$

find $\mathbf{K}$. *Ans.* $\frac{1}{\mu_0}(3.72\mathbf{a}_y - 9.28\mathbf{a}_z)$ (A/m)

13.19. In free space, $\mathbf{D} = D_m \sin(\omega t + \beta z)\mathbf{a}_x$. Using Maxwell's equations, show that

$$\mathbf{B} = \frac{-\omega\mu_0 D_m}{\beta}\sin(\omega t + \beta z)\mathbf{a}_y$$

Sketch the fields at $t = 0$ along the z axis, assuming that $D_m > 0$, $\beta > 0$. *Ans.* See Fig. 13-8

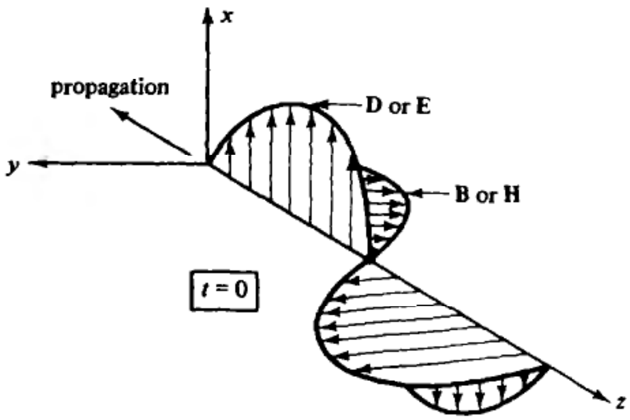

Fig. 13-8

13.20. In free space,

$$\mathbf{B} = B_m e^{j(\omega t + \beta z)} \mathbf{a}_y$$

Show that

$$\mathbf{E} = -\frac{\omega B_m}{\beta} e^{j(\omega t + \beta z)} \mathbf{a}_x$$

13.21. In a homogeneous region where $\mu_r = 1$ and $\epsilon_r = 50$,

$$\mathbf{E} = 20\pi e^{j(\omega t - \beta z)} \mathbf{a}_x \quad (\text{V/m}) \qquad \mathbf{B} = \mu_0 H_m e^{j(\omega t - \beta z)} \mathbf{a}_y \quad (\text{T})$$

Find ω and H_m if the wavelength is 1.78 m. *Ans.* 1.5×10^8 rad/s, 1.18 A/m